1916 Einstein Notation. Einstein summation convention.

MATH 162A Review: Einstein Convention

Facts to Know:

The definition of the Einstein Convention.

$$a_3 + a_4 + \cdots + a_{20} = \cancel{\sum_{j=3}^{20}} a_j = \cancel{\sum_{k=3}^{20}} a_k$$

$$\sum_{j=3}^{20} a_j \sim a_j$$

A $m\times n$ matrix

$$x^T A y = \boxed{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i x_j} \sim a_{ij} x_i x_j$$

① $a_{ij} x_j = b_i \Leftrightarrow$ The equation is true for all i.

② $a_{ij} x_j = b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j = b_i$

For example $\sum_{i=1}^{n} a_i b_i c_i \sim a_i b_i c_i$

$\underline{a_{ij} x_j = b_i} \Leftrightarrow Ax = b$

$$a_{ijk} x_i y_j z_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ijk} x_i x_j x_k$$

$$a_{i_1 \cdots i_s} x_{i_1} \cdots x_{i_s} = \sum \sum$$

tensor algebra

Kronecker symbol $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$AB = I$ $\qquad a_{ij} b_{jk} = \delta_{ik}$

$I = (\delta_{ij})$

Examples:

1. Prove that for any inner product in $\mathbb{R}^n$ there is a positive definite matrix A such that

$$\langle x, y\rangle = x^T Ay.$$

Let $\{e_1 \ldots e_n\}$ be the standard basis.

$x = x_1 e_1 + \cdots + x_n e_n = x_i e_i$

$y = y_i e_i = y_j e_j$

$<x, y> = <x_i e_i, \cancel{y_i e_i}\ y_j e_j> = x_i y_j <e_i, e_j>$

$A = ((e_i, e_j))$

$x^T A x = (e_i, e_j) x_i x_j = (x_i e_i, x_j e_j) = (x, x) \geq 0$

2. Prove the associativity of matrix multipliction.

$$(AB)C = A(BC).$$

$A = (a_{ij})$ $B = (b_{ij})$, $C = (c_{ij})$

$AB = (a_{ij} b_{jk})_{i,k}$

$(AB)C = (a_{ij} b_{jk}) c_{kl} = (a_{ij} b_{jk} c_{kl})_{i,l}$

$BC = (b_{ik} c_{kj})$

$A(BC) = a_{li}(b_{ik} c_{kj}) = (a_{li} b_{ik} c_{kj})_{l,j}$

$l \to i$
$j \to l$
$i \to j$